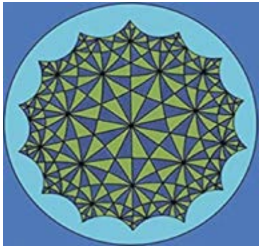
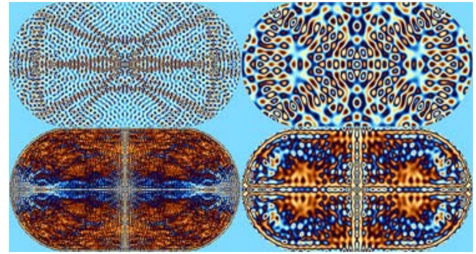


What is... a matrix model?



Raphaël Belliard
HU-Berlin

BMS Days
2/3/2021

Question: Is there such a thing as a common playground for virtually all areas of mathematics?

Where all mathematical preferences can find a place and be inter-related?

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Is this too much to ask?

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Is this too much to ask?

NOT AT ALL!

It exists, and there we find matrix models!

But what are they?

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Short-answer: Families of multiple contour integrals in the complex-plane

But what are they ?

Short-answer : Families of multiple contour integrals in the complex-plane

Thank you for your attention ☺

Origins: Statistical correlation of multi-variate random data.

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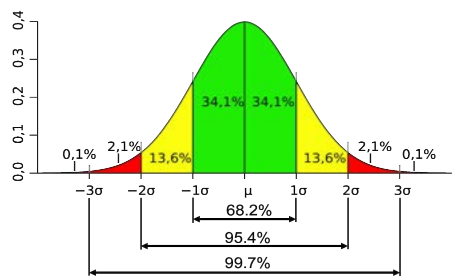
Probability distribution of $M = {}^tX X$

where X matrix of iid Gaussian random vectors



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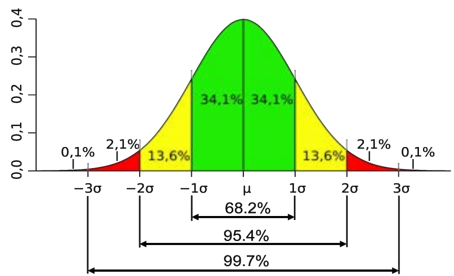

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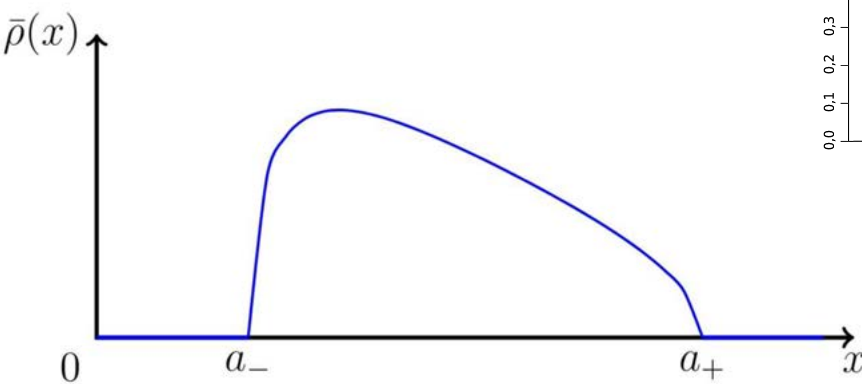

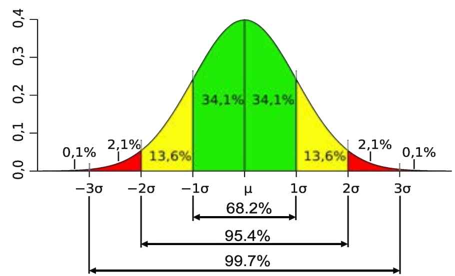

Nuclear physics : Level spacings of heavy-atoms

Multi-particle system
of electrons bound to
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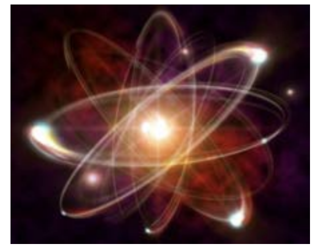

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Concretely: find eigenvalues of linear operators whose ranks grow exponentially with the number of particles.

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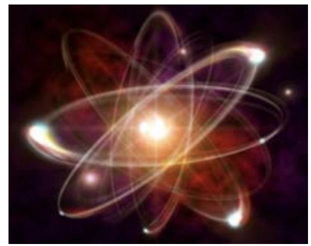

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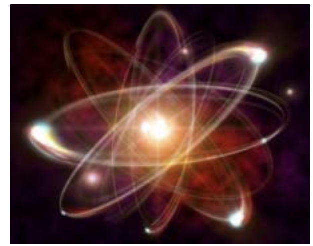

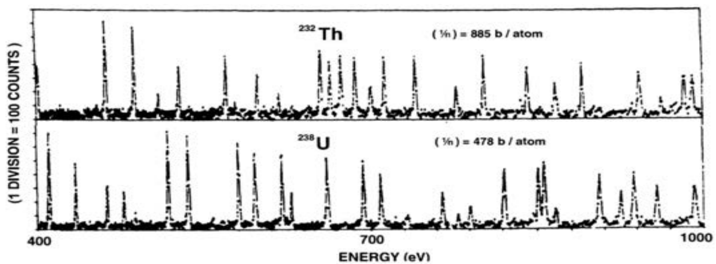

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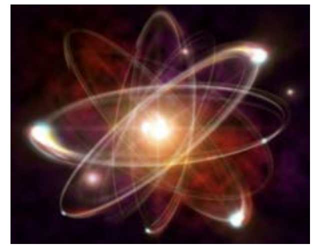

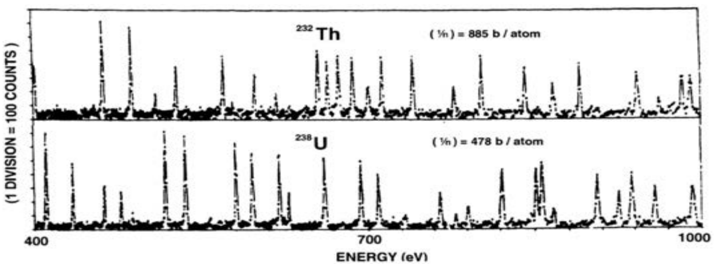

Consecutive eigenvalue spacing distribution recovers experimental level spacings ± 1%

Number theory: Spacing between zeta zeroes

Odlyzko computes 10^5 consecutive zeroes of a certain function on the line $\mathrm{Re}\, s = \frac{1}{2}$ and plots the distribution of their spacings

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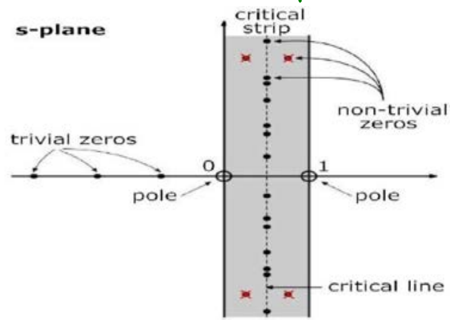
Riemann's Hypothesis



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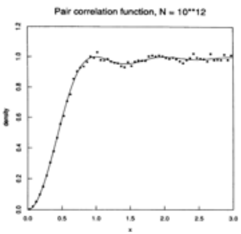

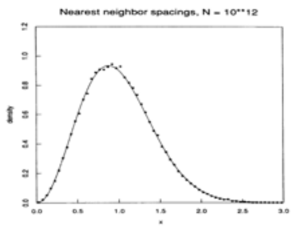

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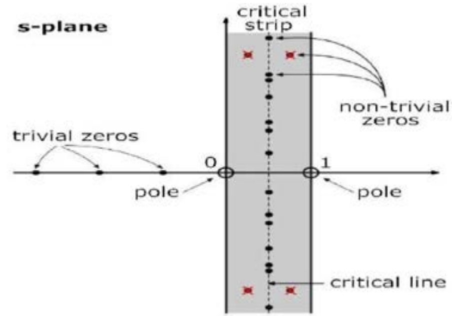

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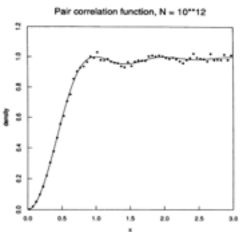

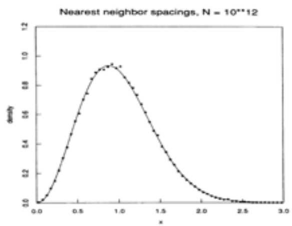

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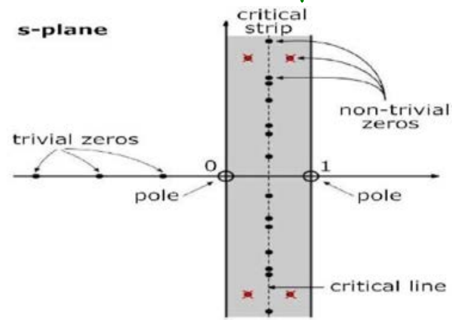

Matches spacing between pairs of eigenvalues of a random unitary matrix

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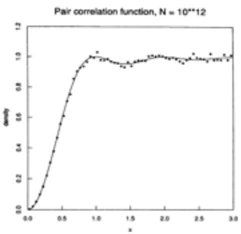

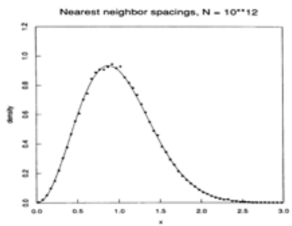

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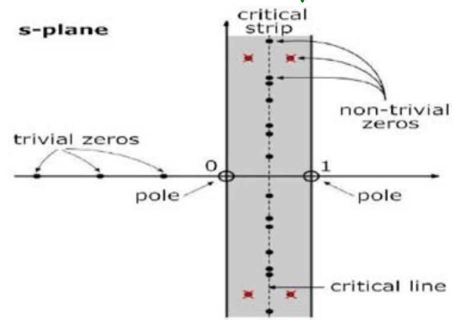

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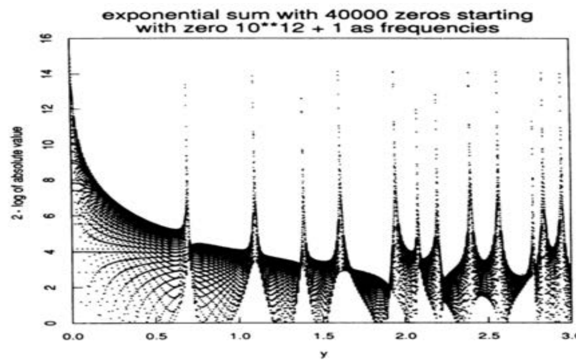

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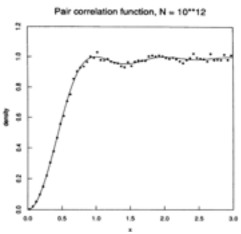

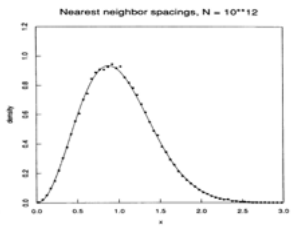

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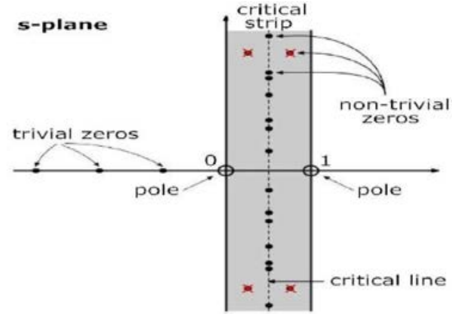

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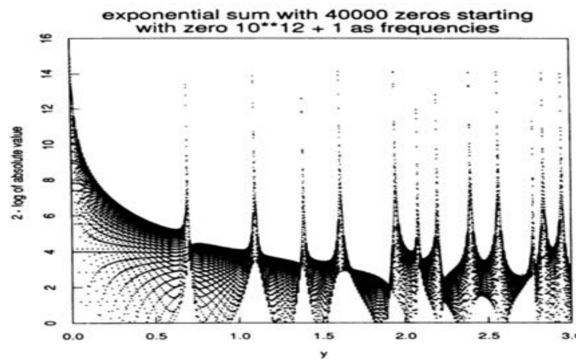
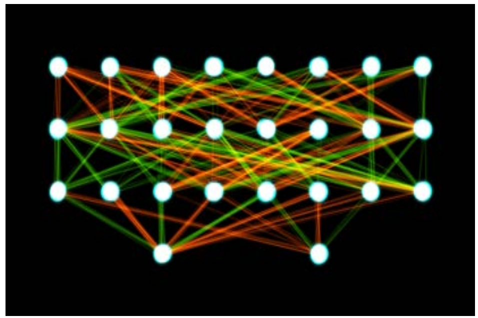


Enough with last century ...

Deep Learning:

Spectral statistics of loss-surface Hessians

Artificial neural networks are complicated graphs on which one minimises a loss-function. The critical points are described by a (Hessian) matrix.



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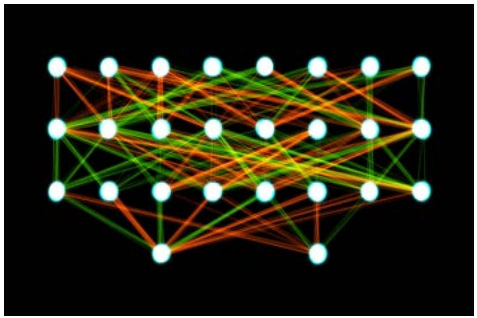

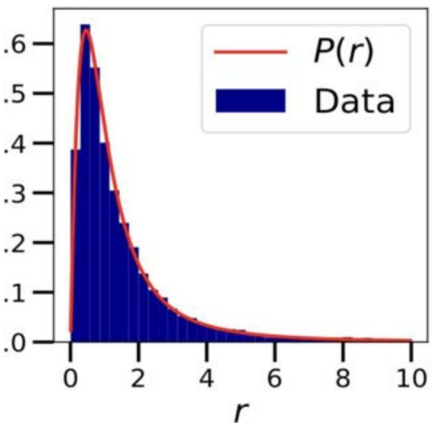

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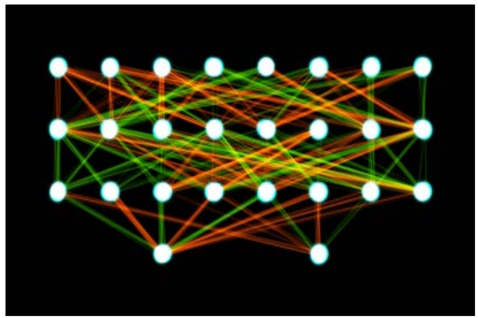
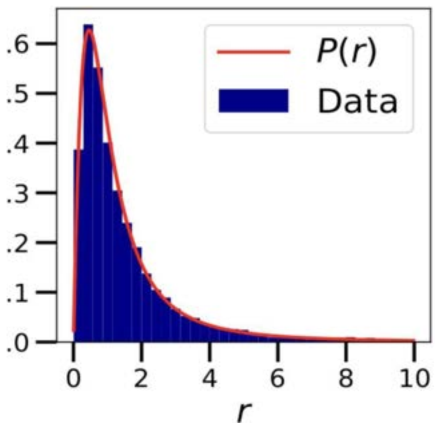
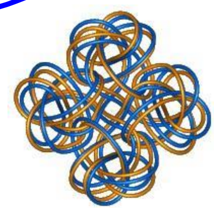




But also: topology
(knots)



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Simplest case: Random hermitian matrix

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$$\mathrm{Prob}(M \in X) = \int_X dM\, p(M)$$

$X \subset \mathcal{H}_N$

probability density

$$\int_{\mathcal{H}_N} dM\, p(M) = 1$$

Basis independance:

A matrix represents an endomorphism in a given basis.

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$$\delta M = \delta U \Lambda U^{-1} + U \delta\Lambda U^{-1} - U \Lambda U^{-1} \delta U U^{-1}$$

$$= U \Big(\underbrace{[U^{-1}\delta U, \Lambda]}_{(*)} + \delta \Lambda \Big) U^{-1}$$

$(*)_{ij} = (U^{-1}\delta U)_{ij} (\lambda_j - \lambda_i)$

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_j - \lambda_i)$$

$$\Longrightarrow dM = \overbrace{\Delta(\Lambda)}^{}{}^2 d_H U \, d\Lambda$$

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* deformed Gaussian

$$\mathrm{Prob}(\Lambda \in X) = \frac{1}{Z_N(V)}\int d\Lambda\, \Delta(\Lambda)^2 e^{-\sum_{i=1}^{N} V(\lambda_i)}$$

$X \subset \mathbb{R}^N$

$$V(\lambda) = \frac{1}{2}\sigma\lambda^2 + \sum_{k=3}^{d} t_k \lambda^k$$

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Partition Function

Now what?

Now what? Now we calculate.

$$Z_N(V) = \int_{\mathbb{R}^N} d\lambda\, \Delta(\lambda)^2 e^{-\sum_{k=1}^{N} V(\lambda_k)}, \quad V(\lambda) = \frac{1}{2}\sigma\lambda^2 + \sum_{k=3}^{d} t_k \lambda^k$$

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* differentiate

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Generalisations: $\underbrace{\mathcal{H}_N, \mathcal{U}_N}_{O_N, Sp_{2N}, \text{etc.}}, \underbrace{\mathbb{R}^N, V}_{\gamma^N \text{ with } \left|\int_\gamma d\lambda\, e^{-V(\lambda)}\right| < \infty}$

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* But also $N \to \infty$

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Integrability: Exact solution to the problem

Dyson 1970's

Orthogonal Polynomials

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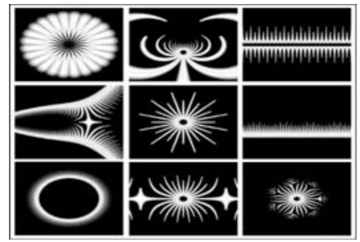

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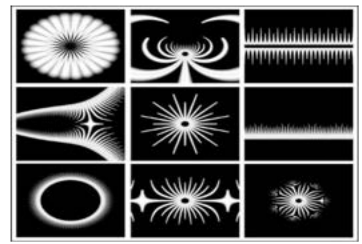
Orthogonal Polynomials

Akkeman, Ambjorn, Chekhov 1990's
Loop Equations



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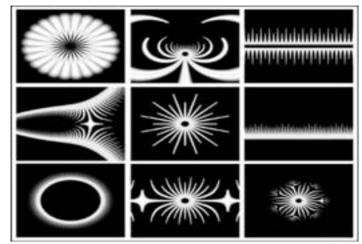
Akkeman, Ambjorn, Chekhov 1990's
Loop Equations

Eynard, Orantin 2000's
Topological Recursion

The analytic methods illustrate the underlying 2D conformal invariance.



Integrability: Exact solution to the problem

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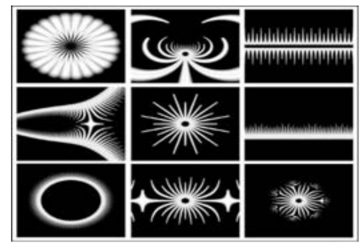

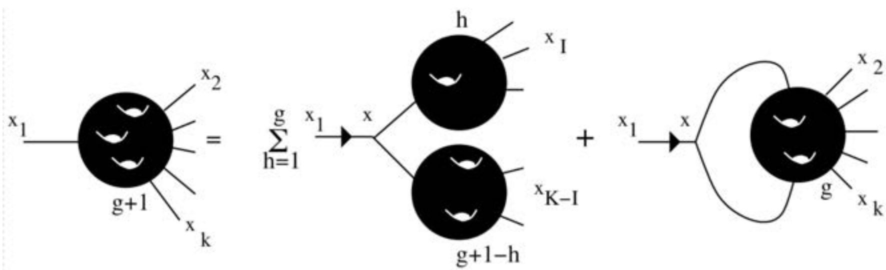

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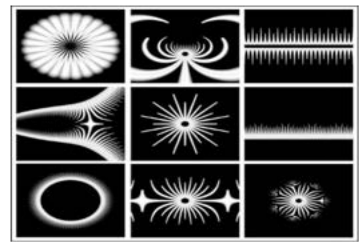

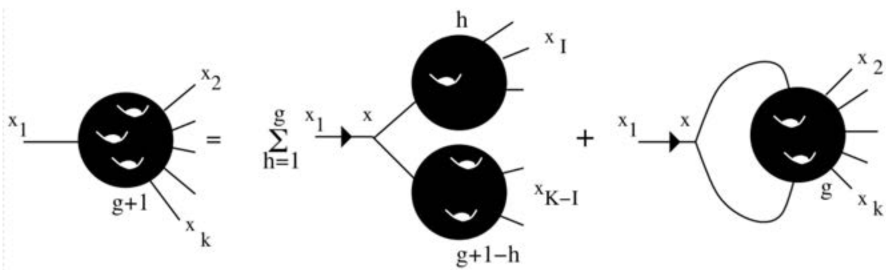

More next talk by Borot